Eur. Phys. J. B **18**, 159–162 (2000)

THE EUROPEAN PHYSICAL JOURNAL B

EDP Sciences

On coordination and continuous hawk-dove games on small-world networks

E. Ahmed[1,2] and A.S. Elgazzar[3,a]

[1] Mathematics Department, Faculty of Science, Al-Ain PO Box 17551, UAE
[2] Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[3] Mathematics Department, Faculty of Education, El-Arish 45111, Egypt

Received 8 July 2000 and Received in final form 23 July 2000

Abstract. It is argued that small-world networks are more suitable than ordinary graphs in modelling the diffusion of a concept (*e.g.* a technology, a disease, a tradition, ...). The coordination game with two strategies is studied on small-world networks, and it is shown that the time needed for a concept to dominate almost all of the network is of order $\log(N)$, where N is the number of vertices. This result is different from regular graphs and from a result obtained by Young. The reason for the difference is explained. Continuous hawk-dove game is defined and a corresponding dynamical system is derived. Its steady state and stability are studied. Replicator dynamics for continuous hawk-dove game is derived without the concept of population. The resulting finite difference equation is studied. Finally continuous hawk-dove is simulated on small-world networks using Nash updating rule. The system is 2-cyclic for all the studied range.

PACS. 64.60.-i General studies of phase transitions

1 Introduction

Hawk-dove (HD) game [1,2] is a two-player game, such that two strategies, hawk (H) and dove (D), are allowed. Assume that v is the value of gain, and c is the cost of the fight. If a H-player plays against a D-player, the H-player's payoff (profit) will be the whole gain v, and the payoff of the D-player will be zero. When two H-players interact, the payoff of each one will be $\frac{1}{2}(v-c)$, but in the case of two D-players, the payoff of each of them will be $\frac{v}{2}$. This is summarized in a matrix called payoff matrix as follows,

$$\begin{array}{ccc} & \mathrm{H} & \mathrm{D} \\ \mathrm{H} & \frac{v-c}{2} & v \\ \mathrm{D} & 0 & \frac{v}{2} \end{array} \quad . \tag{1}$$

Every player tries to improve his (her) payoff. Max-min solution of von Neumann *et al.* [3] is a method to determine which strategy will give the best payoff. The minimum values of the rows of the payoff matrix are calculated. The strategy which gives the maximum of those minimum values is the max-min solution. For $0 < c \le v$, the solution is to adopt hawk policy. But, if $c > v > 0$, the solution is to follow dove policy. Maynard Smith [1] has shown that this solution is not evolutionary stable since a mutant adopting hawk policy will gain so much that it will encourage others to adopt hawk policy. This will continue till the fraction of hawk's P becomes large enough to make the payoffs of both strategies equal. Thus

$$P = \frac{v}{c}\cdot$$

Consider a 1-dimensional chain of players, each player is allowed to change his (her) strategy according to certain updating rules. In this work Nash updating rule was used. It is defined as follows,

Definition 1. If the player's payoff is larger or equal to every one of his (her) nearest neighbors (NN), then preserve the strategy, else follow the strategy of his (her) NN with the highest payoff. If there are more than one neighbor with the highest payoff, choose one of them randomly.

In the above defined game every player may be either H or D. This can not be found in reality. So a continuous HD game [4], where each player is considered a hawk with a certain degree $x \in [0,1]$, and dove of degree $1-x$ is introduced, where $x = 0(1)$ means that the player is a dove (hawk). This is similar to fuzzy set theory.

In most social networks it is observed that people interact within a close circle of acquaintances where most of one's friends are themselves friends of each other. Sometimes a person has a friend away from this circle. When this is modelled as a network then sites are arranged in a circle with some long-range edges which are typically called shortcuts. Each player interacts with both his (her) NN and shortcut neighbors. This network is called

[a] e-mail: elgazzar@mum.mans.eun.eg

small-world network (SWN) [5,6]. So, SWN are more realistic than both regular and random networks.

In this work we study the diffusion of a concept on SWN. Also, CHD game is studied on SWN. The power law-exponential transition observed in financial price data is explained using SWN.

2 Diffusion in social network using SWN

We begin by studying the diffusion of a concept (*e.g.* a technology, a disease, a tradition, etc.) on SWN according to the following rule: Assume that the population occupies the sites (vertices) of a SWN. If at time t, a site has one of its NN (whether a local or a shortcut neighbor) adopting the concept, then the site will adopt it at time $t+1$ and henceforth. It will be show that, in agreement with [7], the number of the concept followers will initially grow as a power law, then it will grow exponentially. Consequently the time needed for the concept to diffuse throughout almost all of the SWN is of order $\log(N)$, where N is the number of vertices in the network. For a regular graph the time needed is a power law of N, thus the diffusion on a SWN is much faster than diffusion on a regular graph. Assume that the speed of the spread of the concept is unity and let φ be the fraction of shortcuts hence 2φ is the density of shortcut-ends in the graph. The number of infected persons will grow initially as a sphere with surface $\Gamma_d t^{d-1}$, where $\Gamma_1 = 2$, $\Gamma_2 = 2\pi$, $\Gamma_3 = 4\pi$ and so on. This is called the primary sphere. Once a shortcut is reached (the probability of such an event is $2\varphi\Gamma_d t^{d-1}$ per time unit) a secondary sphere forms and so on. Hence the total number of infected persons is given by

$$V(t) = \Gamma_d \int_0^t \tau^{d-1}\left[1 + 2\varphi V(t-\tau)\right] \mathrm{d}\tau. \tag{2}$$

Defining $\widetilde{V} = 2\varphi V$, $\widetilde{t} = t[2\varphi\Gamma_d(d-1)!]^{1/d}$, and differentiating d times with respect to $\widetilde{t}$ one gets

$$\frac{\partial^d \widetilde{V}}{\partial \widetilde{t}^d} = 1 + \widetilde{V},$$

whose solution is

$$\widetilde{V}(\widetilde{t}) = \sum_{i=1}^{\infty} \frac{\widetilde{t}^{di}}{(di)!}\,. \tag{3}$$

In 1-dimension (1D), one has $\widetilde{V}(\widetilde{t}) = \exp(\widetilde{t}) - 1$ and in 2D, $\widetilde{V}(\widetilde{t}) = \cosh(\widetilde{t}) - 1$. For $\widetilde{t} < 1$, the number of infected persons grow as a power law $\widetilde{t}^d/d!$, while for $\widetilde{t} > 1$, it grows exponentially. The transition occurs at $\widetilde{t} = 1$, *i.e.* at $t = [2\varphi\Gamma_d(d-1)!]$.

Now consider the coordination game with two strategies A and B, where A represents (without loss of generality) the new concept diffusing on SWN. The payoff matrix is

$$\begin{array}{ccc} & A & B \\ A & a & 0 \\ B & 0 & b \end{array}\quad,$$

where $a > b > 0$. Define $n_A(n_B)$ to be the number of NN and shortcut neighbors of a given player adopting the A (B) strategy, respectively.

Theorem 1. In the above coordination game if $n_A \geq n_B$ for a given A-player, then he (she) will not change his (her) strategy.

Proof. The payoff of the A-player is an_A. If he (she) changes to B-strategy his (her) payoff becomes bn_B. But $n_A \geq n_B$, $a > b$ thus the player will not flip to the B-strategy.

Definition 2. A graph S is r-close knit, if $\forall\ S_1 \subseteq S$, $S_1 \neq \Phi$ then $\frac{e(S_1,S)}{\sum_{i\in S_1} Z_i} \geq r$, where $e(S_1, S)$ is the number of edges joining S_1 and S, and Z_i is the number of NN including shortcuts of site i.

Define

$$Z = \max_i \{Z_i : i \in S\}.$$

Theorem 2. A SWN is r-close knit graph with $r = \frac{1}{Z}$.

Proof. To minimize r in a connected SWN choose S_1 to consist of a single site (say i) at the end of a shortcut and choose S to be the two sites of the shortcut $\{i, j\}$ plus the NN of j. Thus $e(S_1, S) = 1$ and $\sum_{i\in S_1} Z_i = Z_i \leq Z$. Thus $\frac{e(S_1,S)}{\sum_{i\in S_1} Z_i} \geq \frac{1}{Z}$.

Now one can understand why the result that the time needed for the concept to dominate a SWN is proportional to $\log(N)$ differs from the one obtained by Young [8] that the time needed to dominate r-close knit graph is independent of N. In Young's proof, it has been assumed that $\frac{1}{2} > r > \frac{b}{(a+b)}$, where a and b are the nonzero elements of the payoff matrix of the aforementioned coordination game. But in SWN it has been proved that $r = \frac{1}{Z}$, which depends on the topology of the graph and not on the payoff matrix. Therefore Young's result is not valid on SWN. We believe that SWN is more realistic than a graph satisfying Young's criterion.

3 Continuous hawk-dove game and an associated dynamical system

As has been noticed by Wahl and Nowak [4] continuous games are more realistic than ordinary ones. The payoff of a player with degree x playing against a player with degree y is given by,

$$\Pi(x,y) = \frac{1}{2}(v-c)xy + vx(1-y) + \frac{v}{2}(1-x)(1-y). \tag{4}$$

Now what is the function $y(x)$ which maximizes the payoff $\Pi(x, y(x))$, and conversely what is the function $x(y)$ which maximizes $\Pi(y, x(y))$. To obtain $y(x)$, we set $\frac{d\Pi(x,y(x))}{dx} = 0$ and solve the resulting differential equation and finally obtain

$$y(x) = \frac{\frac{v}{c}x + C_1}{x + \frac{v}{c}},$$

where C_1 is a constant of integration. This gives the following dynamical system associated to CHD game

$$y_{t+1} = \frac{\frac{v}{c}x_t + C_1}{x_t + \frac{v}{c}},\ x_{t+1} = \frac{\frac{v}{c}y_t + C_2}{y_t + \frac{v}{c}}\,\cdot \tag{5}$$

To satisfy the constraints $x_t, y_t \in [0,1]\ \forall t$ then

$$\frac{v}{c} \geq C_1,\ C_2 \geq \left(\frac{v}{c}\right)^2. \tag{6}$$

It is interesting to notice that unless the condition $C_1 = C_2 = \left(\frac{v}{c}\right)^2$ is satisfied, the evolutionary stable solution $x_t = y_t = \frac{v}{c}$ is unattainable. A similar situation has been observed before [9]. The steady state of equations (5, 6) are

$$x = \frac{1}{4\frac{v}{c}}[-(C_1 - C_2) + b],$$
$$y = \frac{1}{4\frac{v}{c}}[(C_1 - C_2) + b], \tag{7}$$

where

$$b = \sqrt{(C_1 - C_2)^2 + 8\left(\frac{v}{c}\right)^2 (C_1 + C_2)}.$$

The corresponding eigenvalues are

$$\lambda_\pm = \frac{d}{(x + \frac{v}{c})(y + \frac{v}{c})},$$
$$d = \sqrt{\left(C_1 - \left(\frac{v}{c}\right)^2\right)\left(C_2 - \left(\frac{v}{c}\right)^2\right)}.$$

The steady state (7) is stable if and only if

$$\frac{1}{2}(C_1 + C_2) + \left(\frac{v}{c}\right)^2 + \frac{b}{2} > d.$$

Define $\alpha = C_1 - C_2$, $\beta = C_1 + C_2$ and

$$h(\alpha,\beta) = \frac{1}{2}\beta + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{b}{2} - d.$$

Then stability of equation (7) is equivalent to proving $h(\alpha,\beta) > 0$. It is direct to see that

$$\frac{\partial h}{\partial \alpha} = \frac{\alpha}{(2b)} + \frac{\alpha}{(2d)},$$

since $b > 0$ and $d > 0$ then for $\alpha \geq 0$, h is a strictly increasing function of α *i.e.*

$$h(\alpha,\beta) > h(0,\beta) = 2\left(\frac{v}{c}\right)^2 + \frac{v}{c}\sqrt{2\beta} > 0.$$

Also using $h(\alpha,\beta) = h(-\alpha,\beta)$, then $h(-\alpha,\beta) > 0$. Thus we have

Theorem 3. The system (5) and (6) has a unique steady state (7), which is asymptotically stable.

4 Replicator equation of CHD game

Replicator equation [9,10] is usually associated to population dynamics, where a fraction $x_i \in [0,1]$ of the population adopts strategy i. The evolution of x_i is described by the replicator equation

$$x_i(t+1) = x_i(t)\frac{(Ax(t))_i + C_1}{xAx + C_1}, \tag{8}$$

where A is the payoff matrix of the game and $C_1$is a large positive constant to ensure that both the numerators and denominators of equation (8) are positive.

For continuous games, one can form replicator dynamics without the need of the population concept as follows: Let a player with hawk degree x plays against an opponent with degree y, then the evolution of $x(t)$ and $y(t)$ can be described by

$$x(t+1) = x(t)\frac{\left(A\begin{bmatrix} y \\ 1-y \end{bmatrix}\right)_1 + C_1}{\underline{x}A\underline{y} + C_1},$$
$$y(t+1) = y(t)\frac{\left(A\begin{bmatrix} x \\ 1-x \end{bmatrix}\right)_1 + C_2}{\underline{y}A\underline{x} + C_2}, \tag{9}$$

where $\underline{x} = [x, 1-x]$, $\underline{y} = [y, 1-y]$, and A is the payoff matrix of the hawk-dove game. For the symmetric case $C_1 = C_2$, $x(t) = y(t)$, then

$$x(t+1) = x(t)\frac{\frac{1}{2}(v-c)x(t) + v(1-x(t)) + C_1}{\underline{x}A\underline{x} + C_1}\,\cdot \tag{10}$$

The steady states are $x = 0, 1, \frac{v}{c}$ and only $x = \frac{v}{c}$ is asymptotically stable.

In the previous study full rationality is assumed *i.e.* the players are assumed to know the payoffs both for them and for their opponents and that they do not make mistakes. This is not realistic so in the following bounded rationality will be assumed. Let w $(1 \geq w > 0)$ be a measure of rationality *e.g.* $w = 1$ is a fully rational player and that $1-w$ is the probability of making mistakes. Then discrete replicator equation can be modified into:

$$x_i(t+1) = (1-w)x_i(t) + wx_i(t)\frac{[Ax(t)]_i + C_i}{xAx + C_i}\,\cdot \tag{11}$$

Applying (11) for the symmetric CHD game, one gets the steady states $x = 0, 1, \frac{v}{c}$ (which was expected since bounded rationality, equation (11) does not change steady states) and that $\forall\ 1 \geq w > 0$, then only $x = \frac{v}{c}$ is asymptotically stable.

Finally we have simulated CHD on SWN with $N = 10\,000$, and 5% shortcuts. The games run for 100 000 time steps. Nash updating rule was used. Each player plays against his (her) NN and shortcut ones, if found. We set $v = 1$ and the values of $c \in [1, 10]$ were varied with step 0.01, then the average of the degree of "hawkish" behavior $\overline{x}$ is calculated and compared with the noncontinuous value $\frac{v}{c}$. In this game, $\overline{x}$ decays very slowly than the non-continuous game which decays as $\frac{v}{c}$. The behavior of $\overline{x}$ as a function of c is studied. At the beginning, there is some fluctuations, then the system become 2-cyclic for all the studied range of c. But the two values are very close to each other.

5 SWN and financial price data

Recently [11] a model has been proposed to explain the observed power law-exponential transition of financial price data vs frequency of trading. The main idea is that information about a certain commodity diffuses so that those who have been informed either all buy or all sell. Random graph has been used in their study. It has been argued here that SWN is more realistic in modelling such effect, so we assume that the network connecting the population of potential buyers is SWN, and show that the observed behavior can be explained. Using the comments after equation (3), the cluster (number) of informed persons grows initially as a power law then it grows exponentially *i.e.* the cluster size $s(t) \propto t^{\beta}$ if $t < t_1$, and $s(t) \propto \exp(t)$ if $t > t_1$ for some constant t_1. The price returns $R(t)$ are defined by $R(t) = \ln\left[\frac{p(t)}{p(t-1)}\right]$, where $p(t) = p(t-1)\exp(as(t-1))$, where a is a constant. Thus one has

$$R(t) \propto t^{\beta} \text{ if } t < t_1 \text{ and } R(t) \propto \exp(t) \text{ if } t > t_1. \quad (12)$$

This explains the observed transition of the returns from power law behavior to an exponential behavior.

We thank the referee for his helpful comments.

References

1. J. Maynard Smith, *Evolution and the Theory of Games* (Cambridge Univ. Press, Cambridge, 1982).
2. J. Hofbauer, K. Sigmund, *The Theory of Evolution and Dynamical Systems* (Cambridge Univ. Press, Cambridge, 1996).
3. R. Gibbons, *A Primer in Game Theory* (Simon and Schuster, N.Y. 1992).
4. L.M. Wahl, M. Nowak, J. Theor. Biol. **200**, 307 (1999).
5. D.J. Watts, S.H. Strogatz, Nature **393**, 440 (1998).
6. M.E.J. Newman, D.J. Watts, Phys. Lett. A **263**, 341 (2000).
7. C.F. Moukarzel, Phys. Rev. E **60**, 6263 (1999).
8. P. Young, *Diffusion in Social Network, ICTP Conference on Mathematical Economics*, Trieste 1998.
9. J. Hofbauer, K. Sigmund, *Evolutionary Games and Population Dynamics* (Cambridge University Press, Cambridge, 1998).
10. J. Weibull, *Evolutionary Game Theory* (MIT press, Massachusetts 1995).
11. V.M. Eguiluz, M.G. Zimmermann, *Dispersion of Rumors and Herd Behavior*, `cond-mat/9908069` v.2 (2000).